

A LENGTH FUNCTION FOR WEYL GROUPS OF EXTENDED AFFINE ROOT SYSTEMS OF TYPE A_1

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ABSTRACT. In this work, we study the concept of the “length function” and some of its combinatorial properties for the class of extended affine root systems of type A_1 . We introduce a notion of root basis for these root systems, and using a unique expression of the elements of the Weyl group with respect to a set of generators for the Weyl group, we calculate the length function with respect to a very specific root basis.

0. Introduction

The combinatorial aspects of Weyl groups have always been one of the most important parts of the Lie theory. Among all such aspects, the concept of the “length” for a Weyl group element and its various applications to the whole theory has been of great interest. In this work, we consider the concept of the “length function” and some of its combinatorial properties for the class of extended affine root systems of type A_1 . This is the first attempt in this regard, and we hope our approach offers a model for studying the same concepts for extended affine root systems of other types.

For finite and affine cases, the concept of length, with respect to the so called root bases, is crucial in order to show that the corresponding Weyl groups have the Coxeter presentation [Ca, Chapter 5 and 16]. One knows that in these two cases, the set of simple reflections generate the Weyl group, and that the length of a Weyl group element w is characterized by the number of positive roots mapped to negative roots, by w . Finite and affine root systems are in fact extended affine root systems of nullities 0 and 1, respectively.

In case of extended affine root systems of nullity greater than 1, there is no such notion of root basis (see [Az, Section 5]). Also in this case extended affine Weyl groups are not Coxeter groups [Ho].

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In Section 1, using the definition of affine reflection systems from [AYY], we recall some essential properties of the corresponding Weyl groups of type A_1 from [AN]. An affine reflection system is a generalization of an extended affine root system. In Proposition 1.6, we show that in contrast to extended affine Weyl groups, there is exactly one Weyl group associated to all possible affine reflection systems of type A_1 , in the same ground abelian group. Let R be an affine reflection system of type A_1 in an abelian group A , with Weyl group \mathcal{W} . Let A^0 be the radical of A . We introduce two maps $\varepsilon : \mathcal{W} \rightarrow \{-1, 1\}$ and $T : A^0 \rightarrow A^0$, which will be crucial for the rest of the work. Fixing a finite root system $\tilde{R} = \{0, \pm\epsilon\}$ of type A_1 , in Theorem 1.11, we show that each element $w \in \mathcal{W}$ has a unique expression in the form

$$w = w_\epsilon^{\delta_{\varepsilon(w), 1}} w_{\epsilon+T(w)}.$$

In particular, the Weyl group is isomorphic to the semidirect product of the finite Weyl group of type A_1 and the radical of the form on A .

Section 2 is the core of this work. In this section we consider the so called toroidal root basis \tilde{R} of type A_1 . From definition of \tilde{R} in (1.2), any extended affine root system can be considered as a subset of \tilde{R} . This section has two parts. In the first part, we define a height function for \tilde{R} (Definition 2.1). We also partition \tilde{R} , and so any extended affine root system of type A_1 , into a positive and a negative part (Definition 2.2). Then in Lemma 2.3, we show that a root is positive (resp. negative) if and only if its height is positive (resp. negative). Next, we introduce a notion of root basis (Definition 2.4) and consider the fundamental root basis, the unique root basis whose height of all its elements is one. The second part of this section is dedicated to the calculation of the length of elements of \mathcal{W} with respect to the fundamental root basis, using a combinatorial approach (see Proposition 2.11 and Theorem 2.12). In this part, we also show that for each $w \in \mathcal{W}$, there exists a root α in \tilde{R} such that the length of w with respect to the fundamental root basis is equal to the absolute value of the height of α (see Proposition 2.11 and Corollary 2.13).

Section 3 is dedicated to a discussion on the \mathcal{W} -orbits of root bases. We show that for nullity greater than 1, the number of \mathcal{W} -orbits is not finite. We also calculate the length of each element of \mathcal{W} with respect to any root basis which is a \mathcal{W} -conjugate of the fundamental root basis, see Corollary 3.3.

As mentioned before, the concept of length for elements of the finite and affine Weyl groups are known and calculated. In Section 4, we show that the classical length function, for an affine Weyl group of type A_1 , coincides with the length function described in Section 2. We also show that the classical definition of root basis, in this case, is equivalent to Definition 2.4.

1. Affine reflection systems of type A_1 and their Weyl groups

The class of affine reflection systems was first introduced by E. Neher and O. Loos in [LN]. Here we use an equivalent definition given in [AYY]. Let A be an abelian group and $(\cdot, \cdot) : A \times A \rightarrow \mathbb{Q}$ be a symmetric bi-homomorphism on A , where \mathbb{Q} is the

field of rational numbers. (\cdot, \cdot) is called a form on A . The subgroup

$$A^0 := \{\alpha \in A \mid (\alpha, A) = 0\}$$

of A is called the radical of the form. Set $A^\times := A \setminus A^0$, $\bar{A} := A/A^0$ and let $\bar{\cdot} : A \rightarrow \bar{A}$ be the canonical map. The form (\cdot, \cdot) is called positive definite (positive semidefinite) if $(\alpha, \alpha) > 0$ ($(\alpha, \alpha) \geq 0$) for all $\alpha \in A \setminus \{0\}$. If (\cdot, \cdot) is positive semidefinite, then it is easy to see that

$$A^0 := \{\alpha \in A \mid (\alpha, \alpha) = 0\}.$$

From now on we assume that (\cdot, \cdot) is a positive semidefinite form on A . For a subset B of A , let $B^\times := B \setminus A^0$ and $B^0 := B \cap A^0$. For $\alpha, \beta \in A$, if $(\alpha, \alpha) \neq 0$, set $(\beta, \alpha^\vee) := 2(\beta, \alpha)/(\alpha, \alpha)$ and if $(\alpha, \alpha) = 0$, set $(\beta, \alpha^\vee) := 0$. A subset X of A^\times is called connected if it cannot be written as a disjoint union of two nonempty orthogonal subsets. The form (\cdot, \cdot) induces a unique form on \bar{A} by

$$(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta) \quad \text{for } \alpha, \beta \in A.$$

This form is positive definite on \bar{A} . Thus, \bar{A} is a torsion free group. For a subset S of A , we denote by $\langle S \rangle$, the subgroup generated by S . Here is the definition of an affine reflection system from [AYY].

Definition 1.1. [AYY, Definition 1.3] Let A be an abelian group equipped with a nontrivial symmetric positive semidefinite form (\cdot, \cdot) . Let R be a subset of A . A subset R of A is called an affine reflection system (ARS) in A , if it satisfies the following 3 axioms:

- (R1) $R = -R$,
- (R2) $\langle R \rangle = A$,
- (R3) for $\alpha \in R^\times$ and $\beta \in R$, there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$(\beta + \mathbb{Z}\alpha) \cap R = \{\beta - d\alpha, \dots, \beta + u\alpha\} \quad \text{and} \quad d - u = (\beta, \alpha^\vee).$$

The affine reflection system R is called *irreducible* if it satisfies

- (R4) R^\times is connected.

Moreover, R is called *tame* if

- (R5) $R^0 \subseteq R^\times - R^\times$ (elements of R^0 are non-isolated).

Finally R is called *reduced* if it satisfies

- (R6) $\alpha \in R^\times \Rightarrow 2\alpha \notin R^\times$.

Elements of R^\times (resp. R^0) are called *non-isotropic roots* (resp. *isotropic roots*). An affine reflection system R in A is called a *locally finite root system* if $A^0 = \{0\}$.

Let R be an affine reflection system. Since the form is nontrivial, $A \neq A^0$. The image of R under $\bar{\cdot}$ is shown by \bar{R} . By [AYY, Corollary 1.9], \bar{R} in \bar{A} is a locally finite root system. The *type* and the rank of R are defined to be the type and the rank of \bar{R} , respectively.

Since this work is devoted to the study of Weyl groups of affine reflection systems of type A_1 , for the rest we assume that R is a tame irreducible affine reflection system

of type A_1 . By [AYY, Theorem 1.13], R contains a finite root system $\dot{R} = \{0, \pm\epsilon\}$ of type A_1 and a subset $S \subseteq R^0$, such that

$$R = (S + S) \cup (\dot{R}^\times + S), \quad (1.1)$$

where S is a *pointed reflection subspace* of A^0 , in the sense that it satisfies,

$$0 \in S, \quad S \pm 2S \subseteq S, \quad \text{and} \quad \langle S \rangle = A^0.$$

According to [AYY, Theorem 1.13], any tame irreducible affine reflection system of type A_1 arises in this way. In particular

$$\tilde{R} := A^0 \cup (\dot{R} + A^0) \quad (1.2)$$

is an affine reflection system in A . It follows easily that \tilde{R} contains any affine reflection system of type A_1 in A .

Let $\text{Aut}(A)$ denote the group of automorphisms of A . For $\alpha \in A$, one defines $w_\alpha \in \text{Aut}(A)$ by

$$w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha,$$

for $\beta \in A$. We call w_α the reflection based on α , since it sends α to $-\alpha$ and fixes pointwise the subgroup $\{\beta \in A \mid (\beta, \alpha) = 0\}$ of A . Note that if $\alpha \in A^0$, then according to our convention, $(\beta, \alpha^\vee) = 0$ for all β and so $w_\alpha = \text{id}_A$, where id_A is the identity map on A . For a subset B of R , the subgroup of $\text{Aut}(A)$ generated by w_α , $\alpha \in B$, is denoted by \mathcal{W}_B . The group \mathcal{W}_R is called the Weyl group of R .

Using axiom (R3) of 1.1, we have $w(R) \subseteq R$, for $w \in \mathcal{W}_R$. One can easily conclude that for $w \in \mathcal{W}_R$ and $\alpha, \beta \in A$,

$$(w\alpha, w\beta) = (\alpha, \beta).$$

In turn this leads us to the fact that for $\alpha \in R$ and $w \in \mathcal{W}_R$,

$$ww_\alpha w^{-1} = w_{w(\alpha)}. \quad (1.3)$$

Without loss of generality, we may assume that

$$(\epsilon, \epsilon) = 2.$$

Let $\dot{A} := \langle \dot{R} \rangle$. Then $A = \dot{A} \oplus A^0$, we also have $\langle R^0 \rangle = \langle S \rangle = A^0$. Let $p : A \rightarrow A^0$, be the projections onto A^0 .

For each $\alpha \in A$, we have

$$\alpha = \text{sgn}(\alpha)\epsilon + p(\alpha),$$

where $\text{sgn} : A \rightarrow \mathbb{Z}$ is a group epimorphism. Clearly, each $\alpha \in A$ is uniquely determined by its images under the maps p and sgn . Since the image of p is contained in the radical of the form, the form (\cdot, \cdot) is uniquely determined by the map sgn , namely

$$(\beta, \alpha^\vee) = (\beta, \alpha) = 2\text{sgn}(\beta)\text{sgn}(\alpha),$$

for all $\alpha, \beta \in A$. Here we recall some results from [AN].

Lemma 1.2. [AN, Lemma 2.3] *Let $w := w_{\alpha_1} \cdots w_{\alpha_t} \in \mathcal{W}_R$, $\alpha_i \in R^\times$. Then for $\beta \in R$, we have*

$$\operatorname{sgn}(w\beta) = (-1)^t \operatorname{sgn}(\beta) \quad \text{and} \quad p(w\beta) = p(\beta - 2(-1)^t \operatorname{sgn}(\beta) \sum_{i=1}^t (-1)^i \operatorname{sgn}(\alpha_i) \alpha_i).$$

Proposition 1.3. [AN, Proposition 2.6] *For $\alpha_1, \dots, \alpha_n \in R^\times$, we have $w_{\alpha_1} \cdots w_{\alpha_n} = 1$ in \mathcal{W}_R , if and only if n is even and*

$$\sum_{i=1}^n (-1)^i \operatorname{sgn}(\alpha_i) p(\alpha_i) = 0.$$

In particular, if n is odd, then $w^2 = 1$.

Corollary 1.4. [AN, Corollary 2.8] *For $\alpha, \beta, \gamma \in R$, we have*

$$(w_\alpha w_\beta w_\gamma)^2 = 1.$$

In particular, for $\alpha_1, \dots, \alpha_t \in R^\times$ we have

$$w_{\alpha_1} \cdots w_{\alpha_t} = w_{\alpha_1} \cdots w_{\alpha_{i-1}} w_{\alpha_{i+2}} w_{\alpha_{i+1}} w_{\alpha_i} w_{\alpha_{i+3}} \cdots w_{\alpha_t}.$$

The following definition is suggested by Proposition 1.3.

Definition 1.5. [AN, Definition 2.7] Let P be a subset of R . We call a k -tuple $(\alpha_1, \dots, \alpha_k) \in P$, an *alternating k -tuple in P* if k is even and $\sum_{j=1}^k (-1)^j \operatorname{sgn}(\alpha_j) p(\alpha_j) = 0$. We denote by $\operatorname{Alt}(P)$, the set of all alternating k -tuples in P for all k .

By Proposition 1.3, if $(\alpha_1, \dots, \alpha_k) \in \operatorname{Alt}(P)$, then $w_{\alpha_1} \cdots w_{\alpha_k} = 1$ in \mathcal{W} .

The following proposition is a generalization of [AN, Proposition 4.4] for $\tilde{R} = A^0 \cup (\pm\epsilon + A^0)$.

Proposition 1.6. *We have*

- (i) $w_{\alpha+\sigma+\delta} = w_{\alpha+\sigma} w_\alpha w_{\alpha+\delta}$, for $\alpha \in \tilde{R}$ and $\sigma, \delta \in \tilde{R}^0$.
- (ii) $w_{\alpha+k\sigma} w_\alpha = (w_{\alpha+\sigma} w_\alpha)^k$, $k \in \mathbb{Z}$, $\alpha \in \tilde{R}$ and $\sigma \in \tilde{R}^0$.
- (iii) If Π^0 is any generating subset of A^0 and $\alpha \in \tilde{R}^\times$, then $w_\beta \in \mathcal{W}_{\pm\alpha+\Pi^0}$, for any $\beta \in \tilde{R}^\times = \pm\epsilon + A^0$.
- (iv) $\mathcal{W}_R = \mathcal{W}_{\tilde{R}}$, for any ARS R in A .

Proof. (i)-(ii) The proof is the same as the proof of [AN, Proposition 4.1 (i)-(ii)].

(iii) The proof is essentially the same as the proof of [AN, Proposition 4.1 (iii)], however for the convenience of the reader we provide the details here. Let β be an arbitrary element of $\pm\epsilon + A^0$. Then $\beta = k\epsilon + \sigma$, for $k \in \{\pm 1\}$ and $\sigma \in A^0$. Without loss of generality, we assume that $k = \operatorname{sgn}(\alpha)$. Then we have $\beta = \alpha + \sigma - p(\alpha)$. Let

$$\sigma - p(\alpha) = \sum_{\tau \in \Pi^0} n_\tau \tau,$$

where $n_\tau \in \mathbb{Z}$ and $n_\tau = 0$ for all but a finite number of $\tau \in \Pi_0$. From (i) we have

$$w_\beta w_\alpha = \prod_{\tau \in \Pi^0} w_{\alpha+n_\tau \tau} w_\alpha.$$

Now for each $\tau \in \Pi^0$, from (ii) we have

$$w_{\alpha+n_\tau\tau}w_\alpha = (w_{\alpha+\tau}w_\alpha)^{n_\tau}.$$

This way we obtain an expression of w_β with respect to the reflections based on elements of $\pm\alpha + \Pi^0$. This means that $w_\beta \in \mathcal{W}_{\pm\alpha+\Pi^0}$.

(iv) Let R be an ARS of type A_1 in A . Then $R^\times = \pm\alpha + S$ for a pointed reflection subspace S in A^0 and $\alpha \in \tilde{R}^\times$. Since S generates A^0 , by (iii), we have

$$\mathcal{W}_R = \mathcal{W}_{R^\times} = \mathcal{W}_{\tilde{R}}.$$

□

Suggested by the proof of Proposition 1.6 (iv), we have the following definition.

Definition 1.7. $\mathcal{W} := \mathcal{W}_{\tilde{R}}$ is called the A_1 -type Weyl group on $(A, (\cdot, \cdot))$. If there is no confusion about (\cdot, \cdot) , we simply call \mathcal{W} the A_1 -type Weyl group on A .

In this part we define two maps on \mathcal{W} . These maps are crucial for the rest of this work. For $w \in \mathcal{W} = \langle w_\alpha \mid \alpha \in \tilde{R}^\times \rangle$, suppose that $w_{\alpha_1} \cdots w_{\alpha_n}$ is an expression of w with respect to \tilde{R}^\times , i.e., $\alpha_i \in \tilde{R}^\times$, for $1 \leq i \leq n$. We define $\varepsilon(w) = (-1)^n$. If $w = w_{\beta_1} \cdots w_{\beta_m}$ is another expression of w with respect to \tilde{R}^\times , then from Lemma 1.2 we have

$$(-1)^m \text{sgn}(\gamma) = \text{sgn}(w\gamma) = (-1)^n \text{sgn}(\gamma),$$

for any $\gamma \in \tilde{R}^\times$. Thus the map ε from \mathcal{W} to the multiplicative group $\{-1, 1\}$ is well defined. For $w, w' \in \mathcal{W}$, let $w = w_{\alpha_1} \cdots w_{\alpha_n}$ and $w' = w_{\alpha'_1} \cdots w_{\alpha'_{n'}}$, where $\alpha_i, \alpha'_j \in \tilde{R}^\times$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We have

$$\varepsilon(w w') = (-1)^{n+n'} = \varepsilon(w) \varepsilon(w').$$

So the map $\varepsilon : \mathcal{W} \rightarrow \{-1, 1\}$ is a group homomorphism. For $w \in \mathcal{W}$, we call $\varepsilon(w)$ the sign of w . In this section, we call $w \in \mathcal{W}$ an even (odd) element, if $\varepsilon(w) = 1$ ($\varepsilon(w) = -1$).

Again for $w \in \mathcal{W}$, suppose that $w_{\alpha_1} \cdots w_{\alpha_n}$ is an expression of w with respect to \tilde{R}^\times . We define

$$\tau_w := \sum_{i=1}^n (-1)^{n-i} \text{sgn}(\alpha_i) p(\alpha_i) = \varepsilon(w) \sum_{i=1}^n (-1)^i \text{sgn}(\alpha_i) p(\alpha_i) \in A^0. \quad (1.4)$$

If $w_{\beta_1} \cdots w_{\beta_m}$ is another expression of w with respect to \tilde{R}^\times , then from Lemma 1.2 we have

$$\varepsilon(w) \sum_{j=1}^m (-1)^j \text{sgn}(\beta_j) p(\beta_j) = \frac{1}{2 \text{sgn}(\gamma)} p(\gamma - w\gamma) = \varepsilon(w) \sum_{i=1}^n (-1)^i \text{sgn}(\alpha_i) p(\alpha_i),$$

for $\gamma \in R^\times$. So, τ_w is independent of the choice of the expression for w . Thus the map

$$\begin{aligned} T : \mathcal{W} &\longrightarrow A^0 \\ w &\longmapsto \tau_w \end{aligned}$$

is well defined. Recall that by Proposition 1.6, $w_{\pm\epsilon+\sigma} \in \mathcal{W}$, for $\sigma \in A^0$. Now, since $T(w_{\epsilon+\sigma}) = \sigma$, for $\sigma \in A^0$, T is an onto map. However, T is not one-to-one. To see this, note that $T(w_{-\epsilon+\sigma_1}w_{\epsilon+\sigma_2}) = \sigma_1 + \sigma_2 = T(w_{\epsilon+\sigma_1+\sigma_2})$, but $w_{-\epsilon+\sigma_1}w_{\epsilon+\sigma_2} \neq w_{\epsilon+\sigma_1+\sigma_2}$.

Let $w \in \mathcal{W}$ and $w_{\alpha_1} \dots w_{\alpha_n}$ be an expression of w with respect to \tilde{R}^\times . From Lemma 1.2, we have

$$w(\alpha) = (-1)^n \text{sgn}(\alpha)\epsilon + p(\alpha) - 2\text{sgn}(\alpha) \sum_{i=1}^n (-1)^{n-i} \text{sgn}(\alpha_i)p(\alpha_i).$$

Comparing this equation with definitions of the maps ε and T , we conclude that

$$w(\alpha) = \varepsilon(w)\text{sgn}(\alpha)\epsilon + p(\alpha) - 2\text{sgn}(\alpha)T(w), \quad (1.5)$$

for $\alpha \in \tilde{R}$.

Lemma 1.8. *For $w_1, w_2 \in \mathcal{W}$, we have*

$$T(w_1w_2) = \varepsilon(w_2)T(w_1) + T(w_2).$$

Proof. Let $w_{\alpha_1} \dots w_{\alpha_m}$ and $w_{\beta_1} \dots w_{\beta_n}$ be expressions of w_1 and w_2 with respect to \tilde{R}^\times , respectively. Then

$$\begin{aligned} T(w_1w_2) &= \sum_{i=1}^m (-1)^{m+n-i} \text{sgn}(\alpha_i)p(\alpha_i) + \sum_{j=1}^n (-1)^{m+n-(m+j)} \text{sgn}(\beta_j)p(\beta_j) \\ &= \varepsilon(w_2) \sum_{i=1}^m (-1)^{m-i} \text{sgn}(\alpha_i)p(\alpha_i) + \sum_{j=1}^n (-1)^{n-j} \text{sgn}(\beta_j)p(\beta_j) \\ &= \varepsilon(w_2)T(w_1) + T(w_2). \end{aligned}$$

□

Lemma 1.8 shows that, in general, T is not a group homomorphism.

Proposition 1.9. *The map $w \mapsto \varepsilon(w)(\epsilon + T(w))$ is a one-to-one correspondence from \mathcal{W} onto $\pm\epsilon + A^0$.*

Proof. We define the map $\mathfrak{s} : \mathcal{W} \longrightarrow \pm\epsilon + A^0$ by $\mathfrak{s}(w) = \varepsilon(w)(\epsilon + T(w))$. For $\alpha \in \tilde{R}^\times$, let $w = w_\alpha$ if $\text{sgn}(\alpha) = -1$ and let $w = w_\epsilon w_\alpha$ if $\text{sgn}(\alpha) = 1$. Then $w \in \mathcal{W}$ and $\mathfrak{s}(w) = \alpha$. So \mathfrak{s} is an onto map. On the other hand, let $\mathfrak{s}(w_1) = \mathfrak{s}(w_2)$, for $w_1, w_2 \in \mathcal{W}$. Then $\varepsilon(w_1)\epsilon = \varepsilon(w_2)\epsilon$ and $\varepsilon(w_1)T(w_1) = \varepsilon(w_2)T(w_2)$. So we have $\varepsilon(w_1) = \varepsilon(w_2)$ and $T(w_1) = T(w_2)$. Then by (1.5), For $\alpha \in \pm\epsilon + A^0$, we have

$$\begin{aligned} w_1\alpha &= \varepsilon(w_1)\text{sgn}(\alpha)\epsilon + p(\alpha) - 2\text{sgn}(\alpha)T(w_1) \\ &= \varepsilon(w_2)\text{sgn}(\alpha)\epsilon + p(\alpha) - 2\text{sgn}(\alpha)T(w_2) \\ &= w_2\alpha. \end{aligned}$$

It means that $w_1 = w_2$. Thus \mathfrak{s} is one-to-one. □

Let $\mathcal{W}^0 := \text{Ker}(\varepsilon)$. Then \mathcal{W}^0 is the set of all even elements of \mathcal{W} .

Proposition 1.10. *$T|_{\mathcal{W}^0} : \mathcal{W}^0 \longrightarrow A^0$ is an isomorphism.*

Proof. Let w_1, w_2 be elements of \mathcal{W}^0 . Then by Lemma 1.8

$$T(w_1 w_2) = \varepsilon(w_2)T(w_1) + T(w_2) = T(w_1) + T(w_2).$$

Thus $T|_{\mathcal{W}^0}$ is a homomorphism. If $w \in \text{Ker}(T|_{\mathcal{W}^0})$, then $\varepsilon(w) = 1$ and $T(w) = 0$. Thus by (1.5), $w = 1$. For $\sigma \in A^0$, let $w = w_\epsilon w_{\epsilon+\sigma} \in \mathcal{W}^0$. Then $T(w) = \sigma$. So, $T|_{\mathcal{W}^0}$ is an isomorphism. \square

Theorem 1.11. *Each element w in \mathcal{W} has a unique expression in the form*

$$w = w_\epsilon^{\delta_{\varepsilon(w),1}} w_{\epsilon+T(w)},$$

where δ is the Kronecker delta. In particular, if $\dot{\mathcal{W}} := \langle w_\epsilon \rangle$ is the finite Weyl group of type A_1 , we have

$$\mathcal{W} = \dot{\mathcal{W}} \ltimes \mathcal{W}^0.$$

Proof. To prove the first equation it is enough to show, by Proposition 1.9, that

$$\varepsilon(w) = \varepsilon(w_\epsilon^{\delta_{\varepsilon(w),1}} w_{\epsilon+T(w)}) \quad \text{and} \quad T(w) = T(w_\epsilon^{\delta_{\varepsilon(w),1}} w_{\epsilon+T(w)}).$$

Let $t = \delta_{\varepsilon(w),1}$. By definition of ε , we have $\varepsilon(w_\epsilon^t w_{\epsilon+T(w)}) = -1$, if $\varepsilon(w) = -1$ and $\varepsilon(w_\epsilon^t w_{\epsilon+T(w)}) = 1$, if $\varepsilon(w) = 1$. Thus $\varepsilon(w) = \varepsilon(w_\epsilon^t w_{\epsilon+T(w)})$.

By Lemma 1.8, we have

$$T(w_\epsilon^t w_{\epsilon+T(w)}) = -\delta_{\varepsilon(w),1} 0 + T(w) = T(w).$$

Now, for each $w \in \mathcal{W}$, we have $w = w_\epsilon^{t+1} w_{\epsilon+T(w)}$. Thus $\mathcal{W} = \dot{\mathcal{W}} \mathcal{W}^0$. This together with the facts that $\dot{\mathcal{W}} \cap \mathcal{W}^0 = \{0\}$ and $\mathcal{W}^0 = \text{Ker}(\varepsilon)$ is a normal subgroup of \mathcal{W} show that $\mathcal{W} = \dot{\mathcal{W}} \ltimes \mathcal{W}^0$. \square

Remark 1.12. (i) By Theorem 1.11, every element of \mathcal{W}^0 is of the form $w_\epsilon w_{\epsilon+\sigma}$, for some $\sigma \in A^0$ and by Proposition 1.10, we have $T(w_\epsilon w_{\epsilon+\sigma}) = \sigma$. Let $\varphi : \dot{\mathcal{W}} \rightarrow \text{Aut}(A^0)$ be defined by $\varphi(w_\epsilon^t)(\sigma) = T|_{\mathcal{W}^0}(w_\epsilon^t w_{\epsilon+\sigma} w_\epsilon^t)$. Then it is easy to see that $\mathcal{W} \cong \dot{\mathcal{W}} \ltimes_\varphi A^0$, i.e., \mathcal{W} is isomorphic to the semidirect product of the finite Weyl group of type A_1 and the radical of the form (\cdot, \cdot) on A .

(ii) Let $\mathcal{W}^1 = \{w_\alpha \mid \alpha \in \epsilon + A^0\}$. From Theorem 1.11, we have $\mathcal{W} = \mathcal{W}^0 \uplus \mathcal{W}^1$, where \uplus means disjoint union. Both \mathcal{W}^0 and \mathcal{W}^1 are in one-to-one correspondence with A^0 . Thus one can consider \mathcal{W} as a union of two copies of A^0 .

At the end of this section, using (1.5) and the unique expression of Theorem 1.11, we prove some interesting identities.

Lemma 1.13. *Let $w, w_1, w_2 \in \mathcal{W}$. We have*

- (i) $w^{-1} = w_\epsilon^{\delta_{\varepsilon(w),1}} w_{\epsilon-\varepsilon(w)T(w)} = w_\epsilon^{\delta_{\varepsilon(w),1}} w_{\varepsilon(w)\epsilon-T(w)}.$
- (ii) $w_1 w_2 w_1^{-1} = w_\epsilon^{\delta_{\varepsilon(w_2),1}} w_{\epsilon+\varepsilon(w_1)(T(w_2)+(\varepsilon(w_2)-1)T(w_1))}.$

Proof. (i) Since ε is a group homomorphism, we have $\varepsilon(w^{-1}) = \varepsilon(w)$. Also, from Lemma 1.8, we have

$$0 = T(w w^{-1}) = \varepsilon(w)T(w) + T(w^{-1}).$$

Thus $T(w^{-1}) = -\varepsilon(w)T(w)$. The second equation is obvious.

(ii) Again, it is obvious that $\varepsilon(w_1 w_2 w_1^{-1}) = \varepsilon(w_2)$. From Lemma 1.8 and (i), we have

$$\begin{aligned} T(w_1 w_2 w_1^{-1}) &= \varepsilon(w_1^{-1})T(w_1 w_2) + T(w_1^{-1}) \\ &= \varepsilon(w_1)(\varepsilon(w_2)T(w_1) + T(w_2)) - \varepsilon(w_1)T(w_1) \end{aligned}$$

Thus $T(w_1 w_2 w_1^{-1}) = \varepsilon(w_1)(T(w_2) + (\varepsilon(w_2) - 1)T(w_1))$. From Theorem 1.11, we get the result. \square

2. A length function for the A_1 -type Weyl group \mathcal{W} of nullity ν

In this section we assume that A is a free abelian group of rank $\nu + 1$ and we offer a length function for the A_1 -type Weyl group \mathcal{W} on A . We call \mathcal{W} the A_1 -type Weyl group of nullity ν . Set $\Lambda := A^0$.

Let R be an affine reflection system of type A_1 in A . We identify R with $1 \otimes R$ in

$$\mathcal{V} := \mathbb{R} \otimes_{\mathbb{Q}} A.$$

Then R turns out to be an extended affine root system of type A_1 in \mathcal{V} in the sense of [AABGP, Definition II.2.1]. We simply say that R is an extended affine root system in A . From (1.1), $R = (S + S) \cup (\dot{R}^\times + S)$, where, in this case, the pointed reflection space S is a *semilattice* in Λ in the sense of [AABGP, Definition II.1.2], namely S is a subset of Λ satisfying

$$0 \in S, \quad S \pm 2S \subseteq S, \quad \langle S \rangle = \Lambda.$$

By [AABGP, Remark II.1.6], we have

$$S = \bigcup_{i=0}^m (\tau_i + 2\Lambda),$$

where $\tau_0 = 0$ and τ_i 's represent distinct cosets of 2Λ in Λ , for $0 \leq i \leq m$. The positive integer m is called the index of S . By [AABGP, Proposition II.1.11], Λ has a \mathbb{Z} -basis consisting of elements of S .

We assume that $\mathfrak{B} = \dot{\mathfrak{B}} \cup \mathfrak{B}^0$ is a fixed \mathbb{Z} -basis of A , where $\mathfrak{B}^0 = \{\sigma_1, \dots, \sigma_\nu\}$ is a \mathbb{Z} -basis of Λ and $\dot{\mathfrak{B}} = \{\epsilon\}$ is a \mathbb{Z} -basis of the fixed complement \dot{A} of Λ . The extended affine root system

$$\tilde{R} = \Lambda \cup (\pm\epsilon + \Lambda)$$

in A is called the *toroidal* root system of type A_1 . Let

$$S_b := \bigcup_{i=0}^\nu (\sigma_i + 2\Lambda),$$

where $\sigma_0 = 0$. It is clear that S_b is a semilattice in A^0 . We call the extended affine root system

$$R_b := (S_b + S_b) \cup (\pm\epsilon + S_b)$$

the *baby* extended affine root system of type A_1 . Up to isomorphism, we may assume that for any extended affine root system R of type A_1 in A , we have $\epsilon \in R^\times$ and $\mathfrak{B}^0 \subseteq S$, where S is the semilattice associated with R . Then $R_b \subseteq R \subseteq \tilde{R}$. We have

$$\tilde{R} = \{k\epsilon + \sigma \mid k \in \{0, \pm 1\}, \sigma \in \Lambda\}.$$

For $\sigma = \sum_{i=1}^\nu m_i \sigma_i \in \tilde{R}^0 = \Lambda$, let

$$m^+ := \sum_{\substack{i=1 \\ m_i > 0}}^\nu m_i \quad \text{and} \quad m^- := \sum_{\substack{i=1 \\ m_i < 0}}^\nu m_i, \quad (2.6)$$

where the sum on an empty set is considered to be zero. Then, we define the *height function* $\text{ht}_{\mathfrak{B}^0} : \Lambda \rightarrow \mathbb{Z}$ as follows. For $\sigma \in \Lambda$, let

$$\text{ht}_{\mathfrak{B}^0}(\sigma) := \begin{cases} 2m^+, & m^+ \geq |m^-|, \\ 2m^-, & m^+ < |m^-|. \end{cases} \quad (2.7)$$

Now, we extend the height function to \tilde{R} as follows.

Definition 2.1. For $\alpha = k\epsilon + \sigma \in \tilde{R}$, we define

$$\text{ht}(\alpha) := \begin{cases} k - \text{ht}_{\mathfrak{B}^0}(\sigma), & k = -1, m^+ = |m^-|, \\ k + \text{ht}_{\mathfrak{B}^0}(\sigma), & \text{otherwise.} \end{cases}$$

If $\alpha \in \tilde{R}$ is a nonzero root, then $\text{ht}(\alpha) \neq 0$.

Let $(\cdot, \cdot)_E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ be a form on Λ defined by

$$\left(\sum_{i=1}^\nu m_i \sigma_i, \sum_{i=1}^\nu n_i \sigma_i\right)_E := \sum_{i=1}^\nu m_i n_i,$$

where $m_i, n_i \in \mathbb{Z}$, for $1 \leq i \leq \nu$. If one considers the ν dimensional vector space $\mathcal{V}^0 := \mathbb{R} \otimes_{\mathbb{Q}} \Lambda$ as an Euclidean space, then $(\cdot, \cdot)_E$ is the restriction of the Euclidean inner product on \mathcal{V}^0 to Λ . Using $(\cdot, \cdot)_E$, we define a notion of positive and negative roots on \tilde{R} .

Definition 2.2. Let $\alpha \in \tilde{R} \setminus \{0\}$.

(i) We call α positive, if

$$\text{sgn}(\alpha) \neq -1 \quad \text{and} \quad (p(\alpha), \sum_{i=1}^\nu \sigma_i)_E \geq 0$$

or

$$\text{sgn}(\alpha) = -1 \quad \text{and} \quad (p(\alpha), \sum_{i=1}^\nu \sigma_i)_E > 0.$$

We denote the set of positive roots by \tilde{R}^+ .

(ii) We call a non-zero root α negative, if $\alpha \in \tilde{R} \setminus \tilde{R}^+$. We denote the set of negative roots by \tilde{R}^- .

We have $\tilde{R} = \tilde{R}^+ \cup \{0\} \cup \tilde{R}^-$, $\tilde{R}^\times = \tilde{R}^{\times+} \cup \tilde{R}^{\times-}$ and $\tilde{R}^0 = \tilde{R}^{0+} \cup \{0\} \cup \tilde{R}^{0-}$, where $\tilde{R}^{\times+} := \tilde{R}^\times \cap \tilde{R}^+$, $\tilde{R}^{\times-} := \tilde{R}^\times \cap \tilde{R}^-$, $\tilde{R}^{0+} := \tilde{R}^0 \cap \tilde{R}^+$ and $\tilde{R}^{0-} := \tilde{R}^0 \cap \tilde{R}^-$.

Lemma 2.3. *Let $\alpha \in \tilde{R}$. We have $\alpha \in \tilde{R}^+$ (resp. $\alpha \in \tilde{R}^-$) if and only if $\text{ht}(\alpha) > 0$ (resp. $\text{ht}(\alpha) < 0$).*

Proof. Let $\alpha = \text{sgn}(\alpha)\epsilon + p(\alpha) \in \tilde{R}$ and $p(\alpha) = \sum_{i=1}^{\nu} m_i \sigma_i$. We have

$$(p(\alpha), \sum_{i=1}^{\nu} \sigma_i)_E = \sum_{i=1}^{\nu} m_i = m^+ + m^-,$$

where m^+ and m^- are as in (2.6). Since $\tilde{R}^- \subseteq -\tilde{R}^+$ and $\text{ht}(\alpha) = -\text{ht}(-\alpha)$, for $\alpha \in \tilde{R}^-$, it is enough to show that $\alpha \in \tilde{R}^+$ if and only if $\text{ht}(\alpha) > 0$.

Let $\alpha \in \tilde{R}^+$. From Definition 2.2 (i), if $\text{sgn}(\alpha) \neq -1$, $m^+ + m^- \geq 0$ or if $\text{sgn}(\alpha) = -1$, $m^+ + m^- > 0$. Thus $m^+ \geq -m^-$ or $m^+ > -m^-$. From Definition 2.1, we have $\text{ht}(\alpha) = \text{sgn}(\alpha) + 2m^+ > 0$.

Let α be an element of \tilde{R} for which $\text{ht}(\alpha) > 0$. Thus, from Definition 2.1, we have $\text{ht}(\alpha) = \text{sgn}(\alpha) + \text{ht}(p(\alpha))$ and $\text{ht}(p(\alpha)) \geq 0$. Thus $m^+ \geq -m^-$, if $\text{sgn}(\alpha) \neq -1$ or $m^+ > -m^-$, if $\text{sgn}(\alpha) = -1$. \square

For $\sigma \in \tilde{R}^0 \setminus \{0\}$, let $\sigma = \sum_{i=1}^{\nu} m_i \sigma_i$. We call σ a *strictly positive isotropic root* (resp. *strictly negative isotropic root*), if $m_i \geq 0$ (resp. $m_i \leq 0$), for $1 \leq i \leq \nu$.

Definition 2.4. Let R be an extended affine root system in A and $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu}\}$ be a \mathbb{Z} -basis of A in R_b^{\times} . We call Π a *root basis for R* if each strictly positive isotropic root σ in R can be written in the form $\sigma = \sum_{i=0}^{\nu} m_i \alpha_i$, where $m_i \geq 0$.

Let Π_0 be the set of all elements of \tilde{R} with height 1. We have

$$\Pi_0 = \{\alpha_0 := \epsilon, \alpha_1 := \sigma_1 - \epsilon, \dots, \alpha_{\nu} := \sigma_{\nu} - \epsilon\},$$

where $\mathfrak{B} = \{\epsilon, \sigma_1, \dots, \sigma_{\nu}\}$ is the fixed \mathbb{Z} -basis of A . It is easy to show that Π_0 is a root basis for \tilde{R} . Since the elements of Π_0 are the only elements of \tilde{R} with height 1, we call it *the fundamental root basis for R* . We calculate the length function for \mathcal{W} with respect to the fundamental root basis Π_0 using the height function of Definition 2.1.

Proposition 2.5. *Let $\alpha = \sum_{i=0}^{\nu} n_i \alpha_i \in \tilde{R}$. We have*

$$|\text{ht}(\alpha)| = \sum_{i=0}^{\nu} |n_i|.$$

In particular $\sum_{i=0}^{\nu} |n_i|$ is odd, if $\alpha \in \tilde{R}^{\times}$ and $\sum_{i=0}^{\nu} |n_i|$ is even, if $\alpha \in \tilde{R}^0$.

Proof. Since $\alpha \in \tilde{R}$, we have $\alpha = k\epsilon + \sigma$, where $k \in \{0, \pm 1\}$ and $\sigma \in \Lambda$. Let $\sigma = \sum_{i=1}^{\nu} m_i \sigma_i$. On the other hand, we have $\sigma_i = \alpha_0 + \alpha_i$ and $\epsilon = \alpha_0$. Thus

$$\alpha = (k + \sum_{i=1}^{\nu} m_i) \alpha_0 + \sum_{i=1}^{\nu} m_i \alpha_i.$$

Since Π_0 is a \mathbb{Z} -basis for A , we have $n_0 = k + \sum_{i=1}^{\nu} m_i = k + m^+ + m^-$ and $n_i = m_i$, for $1 \leq i \leq \nu$. We consider three cases.

First we assume that $k = -1$ and $m^+ = |m^-|$. Then $n_0 = -1$. Thus

$$\sum_{i=0}^{\nu} |n_i| = 1 + m^+ + |m^-| = 2m^+ + 1 = -(-2m^+ - 1) = |\text{ht}(\alpha)|.$$

Next we assume that either $m^+ > |m^-|$, or $m^+ = |m^-|$ and $k \neq -1$. Then $m^+ + m^- = m^+ - |m^-| \geq 0$. Thus $n_0 = k + m^+ + m^- \geq 0$. So, we have

$$\sum_{i=0}^{\nu} |n_i| = k + m^+ + m^- + m^+ - m^- = 2m^+ + k = |\text{ht}(\alpha)|.$$

Finally, we assume that $m^+ < |m^-|$. Then $n_0 = k + m^+ + m^- \leq 0$. Thus

$$\sum_{i=0}^{\nu} |n_i| = -k - m^+ - m^- + m^+ - m^- = -2m^- - k = |\text{ht}(\alpha)|.$$

□

Here we need to recall the definition of the length function of an arbitrary group, with respect to some generating subset. Let G be a group and $\{g_\alpha\}_{\alpha \in \Pi}$ be a set of generators for G , namely

$$G = \langle g_\alpha \mid \alpha \in \Pi \rangle.$$

Then we have the following definition.

Definition 2.6. (i) An expression of an element g in $G \setminus \{1\}$, with respect to Π , is a sequence $g_{\alpha_1} \cdots g_{\alpha_k}$ which equals to g , for $\alpha_i \in \Pi$. The positive integer k is called the length of the expression. In this case we say that this expression has occupied k positions, in which g_{α_j} is in the j -th position. A position is called *odd (even)* if j is odd (even).

(ii) The length of $g \in G$ with respect to Π , which is denoted by $\ell_\Pi(g)$, is the smallest length of any expression of g . By convention, $\ell_\Pi(1) = 0$.

(iii) The function ℓ_Π from G to the set of non-negative integers $\mathbb{Z}_{\geq 0}$, which assigns to each $g \in G$, the integer $\ell_\Pi(g)$ is called the length function of G with respect to Π . If there is no confusion about Π , we show the length function by ℓ .

(iv) Any expression of $g \in G$ with length $\ell_\Pi(g)$ is called a reduced expression of g with respect to Π .

Here are some elementary properties of the length function, which only depend on the above definition (See [Hu, 5.2]). For $g, g' \in G$, we have

- $\ell(g) = \ell(g^{-1})$.
- $\ell(g) = 1$ if and only if $g = g_\alpha$, for $\alpha \in \Pi$.
- $\ell(g) - \ell(g') \leq \ell(gg') \leq \ell(g) + \ell(g')$.

Considering an inner automorphism of G , one can see the following elementary result.

Lemma 2.7. *Let g_0 be an element of G , Π' be a set and ϕ be a one to one map from Π onto Π' . For each $\alpha \in \Pi$, let $g_{\phi(\alpha)} := g_0 g_\alpha g_0^{-1}$. Then $\{g_\beta\}_{\beta \in \Pi'}$ is a set of generators for G and, for each $g \in G$, we have*

$$\ell_{\Pi'}(g) = \ell_{\Pi}(g_0^{-1} g g_0).$$

Now, using the concept of the height and the maps ε and T , we offer a formula for the length function for \mathcal{W} with respect to Π_0 . Let $w \in \mathcal{W}$. Until the end of this section, any expression of w is considered with respect to Π_0 . Let $w_{\alpha_{i_1}} \dots w_{\alpha_{i_k}}$ be an expression of w . For $0 \leq i \leq \nu$, we denote the number of w_{α_i} 's for which w_{α_i} is in an odd (resp. even) position by P_i (resp. N_i). We have

$$k = \sum_{i=0}^{\nu} (P_i + N_i). \quad (2.8)$$

Lemma 2.8. *Let $w \in \mathcal{W}$ and $w_{\alpha_{j_1}} \dots w_{\alpha_{j_k}}$ be an expression of w . If $T(w) = \sum_{i=1}^{\nu} m_i \sigma_i$ then*

$$m_i = (-1)^k (P_i - N_i) = \varepsilon(w) (P_i - N_i).$$

Proof. From (1.4), we have

$$\begin{aligned} T(w) &= T(w_{\alpha_{j_1}} \dots w_{\alpha_{j_k}}) \\ &= \sum_{s=1}^k (-1)^{k-s} \text{sgn}(\alpha_{j_s}) p(\alpha_{j_s}). \end{aligned}$$

For $1 \leq i \leq \nu$, let $p_i : \mathcal{V}^0 \rightarrow \mathbb{R}$ be i th coordinate function with respect to \mathfrak{B}^0 . Thus $p_i(\sigma_j) = \delta_{ij}$, where δ is the Kronecker delta function. Then we have

$$\begin{aligned} m_i &= p_i(T(w)) \\ &= \sum_{s=1}^k (-1)^{k-s} \text{sgn}(\alpha_{j_s}) \delta_{ij_s} \\ &= (-1)^k \sum_{s=1}^k (-1)^{s-1} \delta_{ij_s} \\ &= (-1)^k (P_i - N_i). \end{aligned}$$

□

Lemma 2.9. *Let $\sigma = \sum_{i=1}^{\nu} m_i \sigma_i \in \Lambda$ and $w_{\alpha_{j_1}} \dots w_{\alpha_{j_k}}$ be an expression of $w_{\epsilon+\sigma}$.*

- (i) *If $m^+ \geq -m^-$, then $|\text{ht}(\sigma)| \leq k - 1$.*
- (ii) *If $m^+ < -m^-$, then $|\text{ht}(\sigma)| \leq k + 1$.*

Proof. Since $\varepsilon(w_{\epsilon+\sigma}) = -1$, k is odd. Since for every odd position in the expression, except for the last one, there is an even position, we have $\sum_{i=0}^{\nu} P_i = \sum_{i=0}^{\nu} N_i + 1$.

(i) Let J be the set of $0 \leq i \leq \nu$ for which $m_i > 0$. From Definition 2.1 and Lemma 2.8, we have

$$|\text{ht}(\sigma)| = 2m^+ = 2(-1)^k \sum_{i \in J} (P_i - N_i) = 2 \sum_{i \in J} (N_i - P_i) \leq 2 \sum_{i \in J} N_i \leq 2 \sum_{i=0}^{\nu} N_i.$$

Since $\sum_{i=0}^{\nu} P_i = \sum_{i=0}^{\nu} N_i + 1$, using (2.8), we have $2 \sum_{i=0}^{\nu} N_i \leq k - 1$. Thus

$$|\text{ht}(\sigma)| \leq 2 \sum_{i=0}^{\nu} N_i \leq k - 1.$$

(ii) Let J be the set of $0 \leq i \leq \nu$ for which $m_i < 0$. From Definition 2.1 and Lemma 2.8, we have

$$|\text{ht}(\sigma)| = -2m^- = -2(-1)^k \sum_{i \in J} (P_i - N_i) = 2 \sum_{i \in J} (P_i - N_i) \leq 2 \sum_{i \in J} P_i \leq 2 \sum_{i=0}^{\nu} P_i.$$

Since $\sum_{i=0}^{\nu} P_i = \sum_{i=0}^{\nu} N_i + 1$, using (2.8), we have

$$|\text{ht}(\sigma)| \leq 2 \sum_{i=0}^{\nu} P_i = \sum_{i=0}^{\nu} P_i + \sum_{i=0}^{\nu} N_i + 1 \leq k + 1.$$

□

Lemma 2.10. *Let $\sigma = \sum_{i=1}^{\nu} m_i \sigma_i \in \Lambda$ and $w_{\alpha_{j_1}} \cdots w_{\alpha_{j_k}}$ be an expression of $w = w_{\epsilon} w_{\epsilon + \sigma}$. Then we have*

$$|\text{ht}(\sigma)| \leq k.$$

Proof. Since $\varepsilon(w) = 1$, k is even. Since for each odd (even) position in the expression of w there is an even (odd) position, we have $\sum_{i=0}^{\nu} P_i = \sum_{i=0}^{\nu} N_i$. Notice that $T(w) = \sigma$. First let $m^+ \geq -m^-$ and J be the set of $0 \leq i \leq \nu$ for which $m_i > 0$. By Lemma 2.8, using (2.8), we have

$$|\text{ht}(\sigma)| = 2m^+ = 2 \sum_{i \in J} (P_i - N_i) \leq 2 \sum_{i \in J} P_i \leq 2 \sum_{i=0}^{\nu} P_i = k.$$

Next let $m^+ < -m^-$ and J be the set of $0 \leq i \leq \nu$ for which $m_i < 0$. Again, by Lemma 2.8, using (2.8), we have

$$|\text{ht}(\sigma)| = -2m^- = -2 \sum_{i \in J} (P_i - N_i) = 2 \sum_{i \in J} (N_i - P_i) \leq 2 \sum_{i \in J} N_i \leq 2 \sum_{i=0}^{\nu} N_i = k.$$

□

Proposition 2.11. *For $\alpha \in \tilde{R}^{\times}$, we have*

$$\ell_{\Pi_0}(w_{\alpha}) = |\text{ht}(\alpha)|.$$

In particular, if $\alpha = \sum_{i=0}^{\nu} n_i \alpha_i$, we have $\ell_{\Pi_0}(w_{\alpha}) = \sum_{i=0}^{\nu} |n_i|$.

Proof. We have $w_{\alpha} = w_{-\alpha}$ and $|\text{ht}(\alpha)| = |\text{ht}(-\alpha)|$. Thus without loss of generality we may assume that $\text{sgn}(\alpha) = 1$ or equivalently $\alpha = \epsilon + \sigma$, for some $\sigma \in \Lambda$. First we show that w_{α} has an expression of length $|\text{ht}(\alpha)|$ with respect to Π_0 . One should notice that $\varepsilon(w_{\alpha}) = -1$, where ε is the homomorphism defined in Section 1. This means that length of any expression of w_{α} is odd. Also, from Definition 2.1, we have $|\text{ht}(\alpha)|$ is odd. Let

$$\sigma = \sum_{i=1}^{\nu} m_i \sigma_i,$$

as in (2.7) and let J^+ (resp. J^-) be the set of $1 \leq i \leq \nu$ for which $m_i > 0$ (resp. $m_i < 0$). Then we have

$$m^+ = \sum_{i \in J^+} m_i \quad \text{and} \quad m^- = \sum_{i \in J^-} m_i.$$

To build an expression for w_α of required length, we consider two cases $m^+ \geq |m^-|$ and $m^+ < |m^-|$ separately.

Case(i) $m^+ \geq |m^-|$. Let $k := |\text{ht}(\alpha)| = 2m^+ + 1$. We introduce an expression $w_{\beta_1} \cdots w_{\beta_k}$, where $\beta_i \in \Pi_0$, $1 \leq i \leq k$, as follows. Since $k = 2m^+ + 1$, we must have $m^+ + 1$ odd and m^+ even positions in the expression. For each $i \in J^+$, we choose m_i even positions and in each such position r , we put $\beta_r = \alpha_i$. Since the number of even positions is $m^+ = \sum_{i \in J^+} m_i$, there is no even unfilled position in the expression. For each $i \in J^-$, we choose $-m_i$ odd positions and for each such position r , we put $\beta_r = \alpha_i$. Since the number of odd positions which are filled, is $|m^-| = \sum_{i \in J^-} -m_i$ and $|m^-| \leq m^+ < m^+ + 1$, there is at least one unfilled odd position. For any such position r , we put $\beta_r = \alpha_0$.

Case(ii) $m^+ < |m^-|$. Let $k := |\text{ht}(\alpha)| = -2m^- - 1$. Again, we introduce an expression $w_{\beta_1} \cdots w_{\beta_k}$, where $\beta_i \in \Pi_0$, $1 \leq i \leq k$, as follows. Since $k = -2m^- - 1$, we must have $-m^-$ odd and $-m^- - 1$ even positions in the expression. For each $i \in J^-$, we choose $-m_i$ odd positions and in each such position r , we put $\beta_r = \alpha_i$. Since the number of odd positions is $-m^- = \sum_{i \in J^-} -m_i$, there is no odd unfilled position in the expression. For each $i \in J^+$, we choose m_i even positions and for each such position r , we put $\beta_r = \alpha_i$. Up until now, the number of filled even positions is m^+ . On the other hand, $m^+ < |m^-|$ and the total number of even positions in the expression is $|m^-| - 1$. Thus, if $|m^-| \neq m^+ + 1$, there is at least one even unfilled position in the expression. For any such position r , we put $\beta_r = \alpha_0$.

From (1.5), in both cases we have

$$w(\epsilon) = -\epsilon - 2\left(\sum_{i \in J^+} m_i \sigma_i + \sum_{i \in J^-} m_i \sigma_i\right) = -\epsilon - 2\sigma = w_\alpha(\epsilon).$$

Thus we have $w = w_\alpha$, i.e., the given expressions are expressions of w_α in both cases.

Now we show that w_α does not have an expression of smaller length. Assume that w_α has an expression of length p , with respect to Π_0 , where $p < |\text{ht}(\alpha)|$. So $p \leq |\text{ht}(\alpha)| - 2$. If $m^+ \geq -m^-$, then $|\text{ht}(\alpha)| = |\text{ht}(\sigma)| + 1$. Thus $p \leq |\text{ht}(\alpha)| - 2 = |\text{ht}(\sigma)| - 1$. On the other hand, by Lemma 2.9 (i), we have $p \geq |\text{ht}(\sigma)| + 1$. This is a contradiction. If $m^+ < -m^-$, then $|\text{ht}(\alpha)| = |\text{ht}(\sigma)| - 1$. Thus $p \leq |\text{ht}(\alpha)| - 2 = |\text{ht}(\sigma)| - 3$. Again, by Lemma 2.9 (ii), we have $p \geq |\text{ht}(\sigma)| - 1$, which is a contradiction. \square

Recall that according to Theorem 1.11, each element w of \mathcal{W} has an expression of the form $w_\epsilon^{\delta_{\epsilon(w)}, 1} w_{\epsilon+T(w)}$, where ϵ and T are the maps defined in Section 1.

Theorem 2.12. *For $w \in \mathcal{W}$, we have*

$$\ell_{\Pi_0}(w) = |\text{ht}(\epsilon + T(w))| - \text{sgn}(\text{ht}(\epsilon + T(w)))t,$$

where $t = \delta_{\epsilon(w), 1}$, and $\text{sgn}(\text{ht}(\epsilon + T(w)))$ is the sign of $\text{ht}(\epsilon + T(w))$ as an integer.

Proof. By Theorem 1.11, for $t = 0$ we have $w = w_{\epsilon+T(w)}$. This is the case which is discussed in Proposition 2.11. So, let $t = 1$. In this case $\varepsilon(w) = 1$ and therefore $\ell_{\Pi_0}(w)$ is even. Let $w_{\alpha_{i_1}} \dots w_{\alpha_{i_k}}$ be the reduced expression for $w_{\epsilon+T(w)}$ given in the proof of Proposition 2.11. Also, let

$$T(w) = \sum_{i=1}^{\nu} m_i \sigma_i, \quad m^+ := \sum_{\substack{i=1 \\ m_i > 0}}^{\nu} m_i \quad \text{and} \quad m^- := \sum_{\substack{i=1 \\ m_i < 0}}^{\nu} m_i,$$

as in (2.7). Then $w = w_{\epsilon} w_{\alpha_{i_1}} \dots w_{\alpha_{i_k}}$.

If $m^+ \geq |m^-|$ then there is odd $1 \leq j \leq k$ for which $w_{\alpha_{i_j}} = w_{\epsilon}$. Thus in the given expression of w there is at least one w_{ϵ} in an even position. According to Corollary 1.4, we have

$$w = w_{\epsilon} w_{\alpha_{i_2}} \dots w_{\alpha_{i_{j-1}}} w_{\alpha_{i_1}} w_{\alpha_{i_{j+1}}} \dots w_{\alpha_{i_k}} = w_{\alpha_{i_2}} \dots w_{\alpha_{i_{j-1}}} w_{\alpha_{i_1}} w_{\alpha_{i_{j+1}}} \dots w_{\alpha_{i_k}}.$$

Since $w_{\alpha_{i_1}} \dots w_{\alpha_{i_k}}$ is a reduced expression of $w_{\epsilon+T(w)}$, from Corollary 1.4, we have $\alpha_{i_{j-1}} \neq \alpha_{i_1} \neq \alpha_{i_{j+1}}$. Let w has an expression of length p , where $p < |\text{ht}(\epsilon + T(w))| - 1$. Thus $p < |\text{ht}(T(w))|$. This contradicts Lemma 2.10. Thus $\ell_{\Pi_0}(w) = |\text{ht}(\epsilon + T(w))| - 1$.

If $m^+ < |m^-|$ then there is not any w_{ϵ} in an odd position of the reduced expression of $w_{\epsilon+T(w)}$, which is given in Proposition 2.11. In this case w has an expression of length $|\text{ht}(\epsilon + T(w))| + 1$. Let w has an expression of length $p < |\text{ht}(\epsilon + T(w))| + 1$. Thus $p < |\text{ht}(T(w))| - 1 + 1 = |\text{ht}(T(w))|$, which is a contradiction compared with Lemma 2.10. Thus $\ell_{\Pi_0}(w) = |\text{ht}(\epsilon + T(w))| + 1$. \square

Corollary 2.13. *Let $w \in \mathcal{W}^0$. We have*

$$\ell_{\Pi_0}(w) = |\text{ht}(T(w))|.$$

In particular, if $T(w) = \sum_{i=0}^{\nu} n_i \alpha_i$ then $\ell_{\Pi_0}(w) = \sum_{i=0}^{\nu} |n_i|$.

Proof. Since $\mathcal{W}^0 = \text{Ker}(\varepsilon)$, thus $\varepsilon(w) = 1$. By Theorem 1.11, we have $w = w_{\epsilon} w_{\epsilon+T(w)}$. Thus from Theorem 2.12 $\ell_{\Pi_0}(w) = |\text{ht}(\epsilon + T(w))| - \text{sgn}(\text{ht}(\epsilon + T(w)))$. On the other hand, let

$$T(w) = \sum_{i=1}^{\nu} m_i \sigma_i, \quad m^+ := \sum_{\substack{i=1 \\ m_i > 0}}^{\nu} m_i \quad \text{and} \quad m^- := \sum_{\substack{i=1 \\ m_i < 0}}^{\nu} m_i,$$

as in (2.7). We consider the following two cases.

First we assume that $m^+ \geq |m^-|$. By Definition 2.1, $\text{ht}(\epsilon + T(w)) = 2m^+ + 1$. Thus

$$|\text{ht}(\epsilon + T(w))| - \text{sgn}(\text{ht}(\epsilon + T(w))) = 2m^+ + 1 - 1 = 2m^+ = |\text{ht}(T(w))|.$$

Now we assume that $|m^-| > m^+$. In this case $\text{ht}(\epsilon + T(w)) = 2m^- + 1 < 0$. Thus

$$|\text{ht}(\epsilon + T(w))| - \text{sgn}(\text{ht}(\epsilon + T(w))) = -2m^- - 1 + 1 = -2m^- = |\text{ht}(T(w))|.$$

The last assertion in the statement follows from Proposition 2.5. \square

Remark 2.14. The proofs of Proposition 2.11 and Theorem 2.12 give rise to an algorithm for constructing expressions of smallest length for given elements of \mathcal{W} .

3. Orbits and the length function

Recall that A is a free abelian group of rank $\nu + 1$ and $\mathfrak{B} = \{\epsilon, \sigma_1, \dots, \sigma_\nu\}$ is a fixed \mathbb{Z} -basis for A . Let $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_\nu\}$ be a root basis for R . From Definition 2.4, for $1 \leq j \leq \nu$, we have

$$\sigma_j = \sum_{i=0}^{\nu} m_{ji} \alpha_i,$$

where $m_{ji} \geq 0$. For $w \in \mathcal{W}$, let $w(\Pi) = \{w(\alpha_0), w(\alpha_1), \dots, w(\alpha_\nu)\} \subseteq R_b^\times$. Since $w \in \text{Aut}(A)$, $w(\Pi)$ is a \mathbb{Z} -basis of A . Also, We have

$$\sigma_j = w(\sigma_j) = \sum_{i=0}^{\nu} m_{ji} w(\alpha_i) \quad (3.9)$$

Thus $w(\Pi)$ is a root basis for R .

Definition 3.1. Let Π be a root basis for R .

(i) A root basis Π' is called a \mathcal{W} -conjugate of Π , if there exists $w \in \mathcal{W}$ for which $\Pi' = w(\Pi)$. Obviously, conjugacy is an equivalence relation on the set of all root bases for R .

(ii) The \mathcal{W} -orbit of Π is the set of all \mathcal{W} -conjugates of Π .

In Lemma 4.3 it will be shown that a classic root basis of type A_1 , in the usual sense of Lie theory, is a root basis in our terminology. From [Ka, Proposition 5.9], for $\nu = 1$, each root basis of R is a \mathcal{W} -conjugate of Π_0 or $-\Pi_0$. It means that any root basis for R is in one of the two \mathcal{W} -orbits of Π_0 and $-\Pi_0$. This is not true for $\nu > 1$. In fact when $\nu > 1$, the number of \mathcal{W} -orbits is not finite. To see this let $N > 1$ and consider

$$\begin{aligned} \Pi_n = \{ & \beta_0 = \epsilon + 2\sigma_1 + 2\sigma_2, \beta_1 = -\epsilon - \sigma_1 - 2n\sigma_2, \beta_2 = -\epsilon - 2\sigma_1 - \sigma_2, \\ & \beta_3 = -\epsilon - 2\sigma_1 - 2\sigma_2 + \sigma_3, \dots, \beta_\nu = -\epsilon - 2\sigma_1 - 2\sigma_2 + \sigma_\nu \}. \end{aligned}$$

We have $\sigma_1 = (2n-1)\beta_0 + \beta_1 + (2n-2)\beta_2$ and $\sigma_i = \beta_0 + \beta_i$, for $2 \leq i \leq \nu$. It is easy to show that Π_n is a \mathbb{Z} -linearly independent set. Thus Π_n is a root basis for R . Since $\sigma_1 = (2n-1)\beta_0 + \beta_1 + (2n-2)\beta_2$, using (3.9), we conclude that for any $w \in \mathcal{W}$, $\Pi_n \neq w(\Pi_m)$, i.e., Π_m and Π_n are not \mathcal{W} -conjugate, for $m, n > 1$ and $m \neq n$.

Here we show that in each orbit if we know the length function with respect to one basis, it is easy to calculate the length function for other root bases. This is a consequence of Lemma 2.7.

Lemma 3.2. Let Π and Π' be two root bases for R . If there exists $w_0 \in \mathcal{W}$ such that $\Pi' = w_0(\Pi)$, then for $w \in \mathcal{W}$, we have

$$\ell_{\Pi'}(w) = \ell_{\Pi}(w_0^{-1} w w_0).$$

Here we calculate the length function for \mathcal{W} with respect to $w(\Pi_0)$, for $w \in \mathcal{W}$.

Corollary 3.3. Let w_0 be a fixed element of \mathcal{W} . For $w \in \mathcal{W}$, we have

$$\ell_{w_0(\Pi_0)}(w) = |\text{ht}(\epsilon + \tau)| + \text{sgn}(\text{ht}(\epsilon + \tau)) \delta_{\varepsilon(w), 1},$$

where $\tau = \varepsilon(w_0)(T(w) + (\varepsilon(w) - 1)T(w_0))$.

Proof. From Lemma 3.2, we have $\ell_{w_0(\Pi_0)}(w) = \ell_{\Pi_0}(w_0^{-1}ww_0)$. Using Lemma 1.13, we have

$$\varepsilon(w_0^{-1}ww_0) = \varepsilon(w) \quad \text{and} \quad T(w_0^{-1}ww_0) = \varepsilon(w_0)(T(w) + (1 - \varepsilon(w))T(w_0)).$$

Therefore from Theorem 2.12, for $\tau = \varepsilon(w_0)(T(w) + (1 - \varepsilon(w))T(w_0))$, we conclude that

$$\ell_{w_0(\Pi_0)}(w) = |\text{ht}(\epsilon + \tau)| + \text{sgn}(\text{ht}(\epsilon + \tau))\delta_{\varepsilon(w),1}.$$

□

4. Affine Kac-Moody case

In this section, we show that our results about the length function of the Weyl group of a nullity 1 extended affine root system of type A_1 coincide with the results about classical length function on the affine Weyl group, the Weyl group of an affine Kac-Moody root system of type A_1 .

In this section, we assume that $\nu = 1$. Also, as in section 2, we assume that A is a free abelian group of rank $\nu + 1 = 2$. Thus Λ is a free abelian group of rank 1. In this case the only semilattice in Λ is Λ itself. So, there is only one extended affine root system of type A_1 in A , namely

$$\tilde{R} = \Lambda \cup (\pm\epsilon + \Lambda).$$

By identifying \tilde{R} with $\tilde{R} \otimes 1$ as a subset of $\mathcal{V} = \mathbb{R} \otimes_{\mathbb{Q}} A$, \tilde{R} is the affine Kac-Moody root system of type $A_1^{(1)}$ (see [Ka, TABLE Aff1] and [ABGP, Table 1.24]). Thus the A_1 -type Weyl group of nullity 1, which is the Weyl group of \tilde{R} , is the affine Weyl group of type A_1 .

We have

$$\mathcal{V} = \mathbb{R}\epsilon \oplus \mathbb{R}\sigma_1 \quad \text{and} \quad \Lambda = \mathbb{Z}\sigma_1$$

and

$$\tilde{R} = \{k\epsilon + m\sigma_1 \mid k \in \{0, \pm 1\}, m \in \mathbb{Z}\}.$$

Let $\alpha_0 := \epsilon$ and $\alpha_1 := -\epsilon + \sigma_1$. For $\alpha = k\epsilon + m\sigma_1 \in \tilde{R}$, we have $\alpha = (k+m)\alpha_0 + m\alpha_1$. Since $k \in \{0, \pm 1\}$ and $m \in \mathbb{Z}$, if m is positive (resp. negative) then $k+m$ is non-negative (resp. non-positive). Thus, if \uplus is the disjoint union, we have $\tilde{R} = \tilde{R}^+ \uplus \{0\} \uplus \tilde{R}^-$, where

$$\tilde{R}^+ = \{m\alpha_0 + (k+m)\alpha_1 \mid m \geq 0 \text{ if } k = 1, m > 0 \text{ if } k \neq 1\} \quad (4.10)$$

and

$$\tilde{R}^- = \{m\alpha_0 + (k+m)\alpha_1 \mid m \leq 0 \text{ if } k = -1, m < 0 \text{ if } k \neq -1\}. \quad (4.11)$$

\tilde{R}^+ is called the set of positive roots and \tilde{R}^- is called the set of negative roots of \tilde{R} .

Remark 4.1. In the above decomposition of \tilde{R} into the sets of positive and negative roots, we have considered \tilde{R} as an affine Kac-Moody root system and used the classic definition of positive and negative roots. But, using Definition 2.2 of positive and negative roots for extended affine root systems of type A_1 , we could get the same decomposition.

By the proof of [Ka, Proposition 6.5], for each $w \in \mathcal{W}$, there exist unique $n \in \mathbb{Z}$ and $s \in \{0, 1\}$ where

$$w = w_\epsilon^s t_1^n, \quad (4.12)$$

where $t_1 = w_{\alpha_1} w_{\alpha_0}$. As in section 2, let $\Pi_0 = \{\alpha_0, \alpha_1\}$. Then, from [Ka, Exercise 3.6], $\ell_{\Pi_0}(w)$ for each $w \in \mathcal{W}$, is the number of positive roots which are mapped to negative roots by w . Using the above facts one finds a uniform formula for ℓ_{Π_0} as follows.

First, for $n, m \in \mathbb{Z}$ and $k \in \{0, \pm 1\}$, we have

$$(w_{\alpha_1} w_{\alpha_0})^n ((k+m)\alpha_0 + m\alpha_1) = (m - (2n-1)k)\alpha_0 + (m - 2nk)\alpha_1. \quad (4.13)$$

To show this identity, let $\beta = (k+m)\alpha_0 + m\alpha_1$, then $\beta = k\epsilon + m\sigma_1$. By Lemma 1.2, we have

$$\begin{aligned} (w_{\alpha_1} w_{\alpha_0})^n(\beta) &= (-1)^{2n} k\epsilon + m\sigma_1 - 2 \sum_{i=1}^{|n|} (-(-1)^{2n-2i+1} k\sigma_1 + (-1)^{2n-2i} k0) \\ &= k\epsilon + m\sigma_1 - 2nk\sigma_1 \\ &= k\epsilon + (m - 2nk)\sigma_1 \\ &= (m - (2n-1)k)\alpha_0 + (m - 2nk)\alpha_1. \end{aligned}$$

Next, as an easy consequence of (4.13), for $m, n \in \mathbb{Z}$ and $k \in \{0, \pm 1\}$, we have

$$w_{\alpha_0} (w_{\alpha_1} w_{\alpha_0})^n ((k+m)\alpha_0 + m\alpha_1) = (m - (2n+1)k)\alpha_0 + (m - 2nk)\alpha_1, \quad (4.14)$$

Now, we compute the length of elements of \mathcal{W} with respect to Π_0 . For $m \in \mathbb{Z}$ and $k \in \{0, \pm 1\}$, Let $\alpha_{m,k} := (m+k)\alpha_0 + m\alpha_1$.

By (4.13), for $n \in \mathbb{Z}$, we have

$$t_1^n(\alpha_{m,k}) = (m - 2nk)\alpha_0 + (m - 2nk)\alpha_1 = \alpha_{m-2nk,k}.$$

Without loss of generality, we may assume that $n > 0$. Let $\alpha_{m,k} \in R^+$. By (4.10), we have $t_1^n(\alpha_{m,k}) \in R^+$, for $k \in \{-1, 0\}$. For $k = 1$, $t_1^n(\alpha_{m,k}) \in R^-$ if and only if $0 \leq m \leq 2n-1$. Thus $\ell_{\Pi_0}(t_1^n) = 2n$. Since $\ell_{\Pi_0}(w^{-1}) = \ell_{\Pi_0}(w)$, for $w \in \mathcal{W}$, we conclude that

$$\ell_{\Pi_0}(t_1^n) = 2|n|, \quad (n \in \mathbb{Z}). \quad (4.15)$$

By (4.14), we have

$$w_\epsilon t_1^n(\alpha_{m,k}) = (m - (2n+1)k)\alpha_0 + (m - 2nk)\alpha_1 = \alpha_{m-2nk,-k}.$$

First, let $n \geq 0$. By (4.10), we have $w_\epsilon t_1^n(\alpha_{m,k}) \in R^+$ for $k \in \{-1, 0\}$. Also, for $k = 1$, $w_\epsilon t_1^n(\alpha_{m,k}) \in R^-$ if and only if $0 \leq m \leq 2n$. Thus $\ell(w_\epsilon t_1^n) = 2n+1$. Now, let $n < 0$. Again by (4.11), if $k \in \{0, 1\}$ then $w_\epsilon t_1^n(\alpha_{m,k}) \in R^+$. For $k = -1$, we have $w_\epsilon t_1^n(\alpha_{m,k}) \in R^-$ if and only if $m+2n < 0$. It means that $\ell(w_\epsilon t_1^n) = 2|n| - 1$. So, we have

$$\ell(w_\epsilon t_1^n) = |2n+1|, \quad (n \in \mathbb{Z}). \quad (4.16)$$

By combining (4.15) and (4.16), we obtain the following unified formula for the length function ℓ_{Π_0} . For $n \in \mathbb{Z}$ and $s \in \{0, 1\}$,

$$\ell(w_\epsilon^s t_1^n) = 2|n| + \text{sgn}(n)s. \quad (4.17)$$

In Theorem 1.11, we offered a unique expression for elements of the Weyl group of an affine reflection system of type A_1 . For $\nu = 1$, this unique expression is different from the unique expression in (4.12). It also seems that the offered formula for ℓ_{Π_0} in Theorem 2.12 and (4.17) are different. Here, we convert the two unique expressions to each other and show that the given formulas for ℓ_{Π_0} are actually the same.

$$\begin{aligned}
w_\epsilon t_1^n &= w_{\epsilon+n\sigma_1} \implies \ell_{\Pi_0}(w_\epsilon t_1^n) = |\text{ht}(\epsilon + n\sigma_1)| = 2n + 1 = 2n + \text{sgn}(n). \\
w_\epsilon t_1^{-n} &= w_{\epsilon-n\sigma_1} \implies \ell_{\Pi_0}(w_\epsilon t_1^{-n}) = |\text{ht}(\epsilon - n\sigma_1)| = 2n - 1 = 2n + \text{sgn}(n). \\
t_1^n &= w_\epsilon w_{\epsilon+n\sigma_1} \implies \ell_{\Pi_0}(t_1^n) = |\text{ht}(\epsilon + n\sigma_1)| - \text{sgn}(\text{ht}(\epsilon + n\sigma_1)) \\
&= 2n + 1 - 1 = 2n. \\
t_1^{-n} &= w_\epsilon w_{\epsilon-n\sigma_1} \implies \ell_{\Pi_0}(t_1^{-n}) = |\text{ht}(\epsilon - n\sigma_1)| - \text{sgn}(\text{ht}(\epsilon - n\sigma_1)) \\
&= 2n - 1 + 1 = 2n.
\end{aligned}$$

Thus Theorem 2.12 coincides with (4.17) for nullity 1.

In Definition 2.4, we defined a root basis for an extended affine root system of type A_1 . It seems that this definition, for $\nu = 1$, is different from the classical definition of a root basis for an affine Kac-Moody root system. We show that these two definitions are the same in this case. First let us recall the classic definition of a root basis.

Definition 4.2. [Ka, 5.9] Let R be an affine root system of type A_1 . A \mathbb{Z} -basis $\Pi = \{\beta_0, \beta_1\} \subseteq R$ for A is called a classic root basis for R if each root $\alpha \in R$ can be written in the form $\alpha = \pm(n_0\beta_0 + n_1\beta_1)$, where $n_0, n_1 \in \mathbb{Z}_{\geq 0}$.

It is obvious that a classic root basis for R is a root basis with Definition 2.4. Recall from Section 2, that for $\nu = 1$, we have

$$R = \{k\epsilon + n\sigma_1 \mid k \in \{0, \pm 1\}, n \in \mathbb{Z}\}.$$

Lemma 4.3. *Any root basis is a classical root basis.*

Proof. Let $\Pi = \{\beta_0, \beta_1\}$ be a root basis for R . We show that any $\alpha \in R$ can be written in a form $\alpha = \pm(m_0\beta_0 + m_1\beta_1)$, where $m_0, m_1 \geq 0$.

Let $\sigma \in R^0 \setminus \{0\}$. Then $\sigma = n\sigma_1$, for $n \in \mathbb{Z}$. Thus each isotropic root in R is strictly positive or strictly negative. Since Π is a root basis, we have $\sigma = \pm(m_0\beta_0 + m_1\beta_1)$, for $m_0, m_1 \geq 0$. Since $\sigma \neq 0$ is isotropic, we have $m_0, m_1 > 0$. Now we have

$$\pm(m_0\beta_0 + m_1\beta_1) = \sigma = p(\sigma) = \pm(m_0p(\beta_0) + m_1p(\beta_1)).$$

Thus

$$m_0(\beta_0 - p(\beta_0)) + m_1(\beta_1 - p(\beta_1)) = 0.$$

For $i = 0, 1$, we have $\beta_i - p(\beta_i) = \text{sgn}(\beta_i)\epsilon$. Therefore $m_0\text{sgn}(\beta_0) + m_1\text{sgn}(\beta_1) = 0$. Since $\text{sgn}(\beta_0), \text{sgn}(\beta_1) \in \{\pm 1\}$ and $m_0, m_1 > 0$, we have $\text{sgn}(\beta_0)\text{sgn}(\beta_1) = -1$ and $m_0 = m_1$. Without loss of generality, we may assume that $\text{sgn}(\beta_0) = 1$ and $\text{sgn}(\beta_1) = -1$. Now, for $m \in \mathbb{Z}$, we have

$$m\beta_0 + m\beta_1 = mp(\beta_0) + mp(\beta_1) \in R^0.$$

Thus $\sigma_1 = \beta_0 + \beta_1$.

With our assumptions, we have $\epsilon = \beta_0 - p(\beta_0)$. Let $p(\beta_0) = m\beta_0 + m\beta_1$, for $m \in \mathbb{Z}$. Then for $k\epsilon + n\sigma_1 \in R$, where $k \in \{0, \pm\}$ and $n \in \mathbb{Z}$, we can write

$$k\epsilon + n\sigma_1 = (n + k(1 - m))\beta_0 + (n - km)\beta_1.$$

If $n - km > 0$ then $n - km + k > k$ and if $n - km < 0$ then $n - km + k < k$. This completes the proof. \square

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